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Multidimensional Mount Effectiveness for Vibration Isolation

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A method is presented for predicting the performance of multiple mount passive isolation systems. Performance is expressed in terms of a mount effectiveness matrix which relates the structure vibration in an isolated system to the structure vibration in an unisolated, or hard-mounted system. Frequency response functions of the engine, isolator, and structure are used to construct the effectiveness matrix at each frequency of interest. Effectiveness matrix expressions are derived for a mounting system connected at multiple points by massless isolators. Isolator design guidelines are developed for the single and multiple isolator mounting systems. It is shown that a multiple isolator mounting system is effective when the maximum singular value of the isolator impedance matrix is less than the minimum singular value of the equivalent engine and structure impedance matrix. An analytical example shows that the traditional single isolator expression for mount performance does not provide an accurate metric of mount performance for highly coupled multiple mount systems. Results on a simple experiment show good agreement between the predicted and measured structural response using the mount effectiveness.

Introduction

VIBRATION isolation systems have long been used in a variety of applications—from avionics to fixed wing aircraft, in automobiles, etc. In general, vibration isolation systems consist of a vibrating machine, such as an engine, connected to a structure at multiple points by vibration isolators. The purpose of the isolators is twofold: 1) to hold the engine and structure together at a minimal relative displacement at low frequencies, and 2) to reduce structure vibration at high frequencies. Typically, transmissibility has been used to quantify the degree of vibration isolation provided by a mounting system. However, as Sykes[1] and Snowdon[2] have noted, when a mounting system is used to isolate a structure that is nonrigid in the frequency range of interest, transmissibility can erroneously predict excellent mount performance. This is often true at the higher acoustic frequencies, where many authors have reported that it is rare to get more than 20 db of attenuation with isolation mounts, and it is common to get no attenuation at all.

To accurately predict the performance of a mounting system, the dynamic response of the engine, isolators, and structure must be used in the system model. One method, called mount effectiveness, relies on frequency response functions of the uncoupled engine, isolator, and structure to predict the performance of the coupled mounting system. Mount effectiveness was first introduced in the 1950s, and has been described in detail.[1–3] The effectiveness of a mounting system is a nondimensional measure of vibration reduction defined as the ratio of the structure vibration of the hard-mounted (unisolated) system to the structure vibration of the isolated system at the mounting locations. In the context of this article, structure vibration can either be a structural motion (displacement, velocity, or acceleration), or a transmitted force applied to the structure. This method avoids the need to identify the actual vibration disturbance, instead concentrating on a nondimensional comparison between an isolated mounting system and an unisolated mounting system.

Traditionally, the performance of a mounting system has been investigated for a single isolator.[4] In practice, however, engines are attached to structures through multipoint isolation systems. Also, a single mount can provide isolation in multiple directions and, therefore, can be considered to be a system of unidirectional mounts. In these more complex scenarios, single isolator effectiveness computations are not applicable because of the coupling effects in the engine and structure between the various mounting locations. In this article, mount effectiveness is extended to multiple isolator (multidimensional) mounting systems. Also, matrix norms are used to extend the basic concepts developed for the single isolator effectiveness to the multiple isolator case. An analytical example and simple experiment are used to demonstrate and validate the method.

Effectiveness Derivation

In developing a model that is useful for analyzing general vibration isolation problems, frequency response functions (FRFs), characterized by output to input ratios in the frequency domain, are used to represent the dynamic behavior of the engine, isolator, and structure. FRFs are based on the observation that if an object is linear and elastic, a sinusoidal input excitation produces a steady-state sinusoidal output at exactly the same frequency, but with a different magnitude and phase. The magnitude change and phase shift of the output to the input characterize the dynamic (modal) nature of the object in the frequency domain. Analytical models, finite element analyses, or experimental measurements can be used to obtain the FRFs.

Depending on the choice of input and output variables, different frequency response functions can be defined. In vibration analysis, the input and output quantities are forces and motions, where the motion variable is a displacement, velocity, or acceleration. For example, impedance relates a force output to a velocity input. Accelerance, on the other hand, relates an acceleration output to a force input. The nomenclature of the various frequency response functions used in vibration analysis is given by Inman.[5] To provide continuity with earlier single isolator effectiveness derivations, the engine, isolator, and structure dynamic response will be represented by impedances. For a more in-depth description of mechanical impedance, see Ref. 6.

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In addition to modeling the passive dynamics of the engine, isolator, and structure as frequency response functions, the vibration disturbance in the engine is modeled as an equivalent blocked force at the mounting locations. The blocked force is equal to the force generated at the mounting points when the engine is attached to a body of infinite impedance (the engine velocity is zero at the mounting points). In mechanical network theory, the combination of the engine impedance and the equivalent blocked force is called a mechanical Thévenin source.[1] The mechanical Thévenin source is simply a mechanical analogy of the Thévenin equivalent system commonly used in electrical engineering, where the mechanical quantities of force and velocity are respectively substituted for the electrical quantities of voltage and current. This equivalent system formulation is desirable because the actual disturbance, which could be a mass unbalance, torque, etc., is typically not measurable. However, because the effectiveness method relates the structure vibration at the mounting points in the isolated and unisolated systems, knowledge of the exact nature of the disturbance is not needed.

Hard-Mounted (Unisolated) System

A mounting system where the engine is hard-mounted (rigidly attached) to the structure is shown in Fig. 1. The $(n \times 1)$ complex vector f_e^b is an equivalent force representation of the vibratory disturbance at the n mounting points

$$f_e^b = [f_e^{b1} f_e^{b2} \ldots f_e^{bn}]^T \tag{1}$$

where T represents the matrix transpose operation. Each element in the blocked force vector is a complex quantity that varies with frequency. This is also true for all other vectors and matrices defined hereafter. However, for simplicity and convenience, the frequency dependence of each variable will not be shown.

The engine velocities for the unisolated system at the mounting points are represented by the $(n \times 1)$ complex vector v_e^h:

$$v_e^h = [v_e^{h1} v_e^{h2} \ldots v_e^{hn}]^T \tag{2}$$

The structure velocities for the unisolated system at the mounting points are represented by the $(n \times 1)$ complex vector v_s^h:

$$v_s^h = [v_s^{h1} v_s^{h2} \ldots v_s^{hn}]^T \tag{3}$$

Because the engine is rigidly connected to the structure at the mounting locations, the engine and structure velocities are equal at the mounting locations:

$$v_e^h = v_s^h \tag{4}$$

The forces applied by the engine to the structure in the unisolated system at the mounting locations are represented by the $(n \times 1)$ complex vector f^h:

$$f^h = [f^{h1} f^{h2} \ldots f^{hn}]^T \tag{5}$$

The force transmitted by the engine to the structure (f^h) is equal to the vibratory disturbance force f_e^b and the opposing force generated by the engine motion $Z_e v_e^h$

$$f^h = f_e^b - Z_e v_e^h \tag{6}$$

where Z_e is the $(n \times n)$ complex impedance matrix of the engine. The engine impedance matrix is equal to

$$Z_e = \begin{bmatrix} Z_{e11} & Z_{e12} & \cdots & Z_{e1n} \\ Z_{e21} & Z_{e22} & \cdots & Z_{e2n} \\ \cdots & \cdots & \cdots & \cdots \\ Z_{en1} & Z_{en2} & \cdots & Z_{enn} \end{bmatrix} \tag{7}$$

Substituting v_s^h from Eq. (4) into v_e^h from Eq. (6) results in

$$f^h = f_e^b - Z_e v_s^h \tag{8}$$

The structure dynamics can be represented by

$$f^h = Z_s v_s^h \tag{9}$$

where Z_s is the $(n \times n)$ complex impedance matrix of the structure. The structure impedance is equal to

$$Z_s = \begin{bmatrix} Z_{s11} & Z_{s12} & \cdots & Z_{s1n} \\ Z_{s21} & Z_{s22} & \cdots & Z_{s2n} \\ \cdots & \cdots & \cdots & \cdots \\ Z_{sn1} & Z_{sn2} & \cdots & Z_{snn} \end{bmatrix} \tag{10}$$

We can eliminate the unisolated transmitted force f^h and solve for the disturbance force f_e^b in terms of the unisolated structure velocities v_s^h by equating Eqs. (8) and (9):

$$f_e^b = (Z_e + Z_s) v_s^h \tag{11}$$

In the next section, we will develop an expression for the isolated structure velocities in terms of the disturbance force f_e^b. This expression, combined with Eq. (11), will enable us to define an effectiveness matrix relating the isolated and unisolated structure velocities for an identical vibration disturbance.

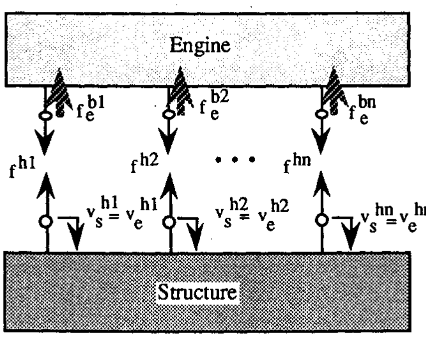

Fig. 1 Hard-mounted (unisolated) system.

Isolated Mounting System

An isolated mounting system is shown in Fig. 2. The vibratory disturbance force is represented by the $(n \times 1)$ complex vector f_e^b. As before, f_e^b is the equivalent disturbance force at the mounting locations when the engine is attached to a body of infinite impedance.

The engine velocities for the isolated system at the mounting points are represented by the $(n \times 1)$ complex vector v_e^i:

$$v_e^i = [v_e^{i1} v_e^{i2} \ldots v_e^{in}]^T \tag{12}$$

The structure velocities for the isolated system at the mounting points are represented by the $(n \times 1)$ complex vector v_s^i:

$$v_s^i = [v_s^{i1} v_s^{i2} \ldots v_s^{in}]^T \tag{13}$$

The forces applied by the engine to the isolators in the isolated system at the mounting locations are represented by the $(n \times 1)$ complex vector f^i:

$$f^i = [f^{i1} f^{i2} \ldots f^{in}]^T \tag{14}$$

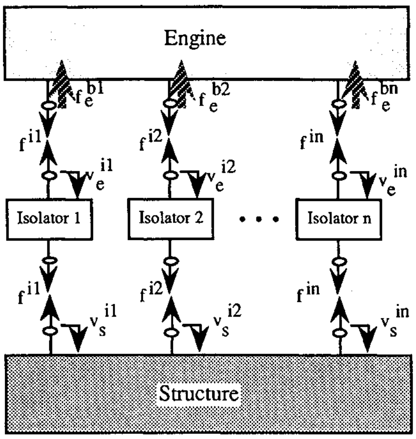

Fig. 2 Isolated mounting system.

The force f_i transmitted from the engine to the isolators is equal to

$$f^i = f_e^b - Z_e v_e^i \tag{15}$$

Equation (15) is similar in form to Eq. (6). However, the isolated transmitted force is different from the unisolated transmitted force because the isolated and unisolated engine velocities are different.

Since the isolators are assumed to be massless, the force transmitted to the engine is equal to the force applied by the engine to the isolator. This analysis can be extended to include isolators with mass effects. The transmitted force across the isolators is given by

$$f^i = Z_i(v_e^i - v_s^i) \tag{16}$$

where Z_i is the $(n \times n)$ impedance matrix of the isolators. Typically, the isolator impedance matrix is a block-diagonal matrix of the form

$$Z_i = \begin{bmatrix} Z_{i1} & 0 & \cdots & 0 \\ 0 & Z_{i2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & Z_{in} \end{bmatrix} \tag{17}$$

The structure dynamics are represented by

$$f^i = Z_s v_s^i \tag{18}$$

By eliminating the transmitted mount force f^i and the engine velocity v_e^i from Eqs. (15), (16), and (18), the structure velocity v_s^i can be expressed solely in terms of the blocked force f_e^b. The engine velocity can be eliminated by solving Eq. (16) for the engine velocity, and substituting the result into Eq. (15). The transmitted mount force can then be eliminated from the resulting expression by substituting Eq. (18) into it. This results in the following expression for the structure velocity:

$$v_s^i = [Z_e + Z_s + Z_e Z_i^{-1} Z_s]^{-1} f_e^b \tag{19}$$

Using Eq. (11), the disturbance force can be eliminated and the unisolated (hard-mounted) structure velocity vector can be related to the isolated structure velocity vector through

$$v_s^h = (Z_e + Z_s)^{-1}[Z_e + Z_s + Z_e Z_i^{-1} Z_s] v_s^i \tag{20}$$

Equation (20) can be simplified to

$$v_s^h = [I + (Z_e + Z_s)^{-1} Z_e Z_i^{-1} Z_s] v_s^i \tag{21}$$

where I is an $(n \times n)$ identity matrix. Equation (21) relates the velocities at the mounting points in the hard-mounted and isolated systems for an identical vibration disturbance f_e^b. From Eq. (21), we can define a matrix E_v that represents the level of velocity effectiveness in the isolated mounting system

$$v_s^h = E_v v_s^i \tag{22}$$

where the velocity effectiveness matrix E_v is

$$E_v = I + (Z_e + Z_s)^{-1} Z_e Z_i^{-1} Z_s \tag{23}$$

Performance with regard to force effectiveness can be derived by inserting Eqs. (9) and (18) into Eq. (22). After some minor simplification, this leads to

$$f^h = Z_s E_v Z_s^{-1} f^i = E_f f^i \tag{24}$$

The force effectiveness matrix E_f is then

$$E_f = I + Z_s(Z_e + Z_s)^{-1} Z_e Z_i^{-1} \tag{25}$$

From Eq. (25), we can define an equivalent series engine and structure impedance matrix as

$$Z_{\text{eq}} = Z_s(Z_e + Z_s)^{-1} Z_e \tag{26}$$

Substituting Eq. (26) into Eq. (25) results in

$$E_f = I + Z_{\text{eq}} Z_i^{-1} \tag{27}$$

The velocity effectiveness matrix defined in Eq. (23) and the force effectiveness matrix defined in Eq. (25) can become difficult to compute if the isolator impedance matrix Z_i and the engine and structure impedance matrix $(Z_e + Z_s)$ are ill-conditioned. In general, structural impedance matrices tend to become ill-conditioned near lightly damped resonant frequencies. The isolator impedance matrix is generally well-conditioned because the isolators possess no resonances (because the isolators are assumed to contain no mass, only stiffness and damping, in this analysis). The matrix $(Z_e + Z_s)$ can become ill-conditioned at a lightly damped resonant frequency of the engine and structure. However, if an ill-conditioning does occur, then the pseudoinverse can be used to closely approximate the inverse of $(Z_e + Z_s)$. This approach has been outlined by Leuridan and Aquilina.[7]

Using Effectiveness for Isolator Design

The effectiveness criteria defined in Eq. (23) for velocity isolation, and Eq. (27) for force isolation, can be used to quantify the performance level of a particular vibration isolation system given the engine, structure, and isolator impedance. This is extremely useful in the analysis stage when the isolator impedances are known. In the design stage, however, the engine and structure impedances are usually given along with a set of performance specifications. In this situation, we need to determine how to design an isolator impedance to achieve a specified performance level. The isolator design problem will be explored in this section for both the single isolator and the multiple isolator mounting systems.

Single Isolator Mounting Systems

As stated in the introduction, the purpose of isolators is twofold: 1) to hold the engine and structure together at a minimal relative displacement at low frequencies, and 2) to reduce structure vibration at high frequencies. Using Eqs. (22) and (24), we can express these two tradeoffs in terms of effectiveness. From Eq. (22), we see that the structure and engine velocities, and hence, displacements, are equal when the magnitude of the effectiveness is equal to unity (the isolator behaves as a hard-mount). On the other hand, Eqs. (22)

and (24) can be used to show that the structure vibration is small in the isolated system compared to the hard-mounted system when the magnitude of the effectiveness is much greater than unity.

When designing isolation systems, we need to determine how to create an isolator that will result in a specified effectiveness. This can be achieved by combining Eqs. (23), (26), and (27) to yield the velocity and force effectiveness for a single isolator mounting system:

$$E_f = E_v = 1 + (Z_{eq}/Z_i) \tag{28}$$

From Eq. (28), we see that the force and velocity (motion) effectiveness quantities are equal for the single isolator mounting system. Also, to achieve an effectiveness magnitude that is approximately equal to one at low frequencies, the isolator impedance must be much greater than the equivalent engine and structure impedance:

$$|Z_i| >> |Z_{eq}| \tag{29}$$

Equation (28) can also be used to show that the effectiveness magnitude is much greater than unity only when the isolator impedance is much less than the equivalent series impedance of the engine and structure:

$$|Z_i| << |Z_{eq}| \tag{30}$$

These concepts can be demonstrated using the standard textbook problem where the isolation system consists of a mass (the engine) and a spring (the isolator) connected to a body of infinite impedance (the structure). For this system, the effectiveness is equal to

$$|E_f| = |E_v| = |1 - (\omega^2/\omega_n^2)| \tag{31}$$

where ω is the forcing frequency and $\omega_n = \sqrt{k/m}$ is the natural frequency of the mounting system.

From Eq. (31), we see that the effectiveness magnitude is equal to unity when the forcing frequency is much less than the natural frequency of the mounting system. Isolation occurs when the driving frequency is much greater than the natural frequency of the mounting system. Of course, for real systems this idealized performance is never reached at high frequencies because the engine does not behave as a mass, and the structure is not rigid. Both the engine and structure typically contain many resonances and antiresonances in the bandwidth of interest. Equation (30) can be used to demonstrate that isolation is degraded at an engine-structure resonance (because Z_{eq} is very small at a resonance) and improved at an antiresonance (because Z_{eq} is very large at an antiresonance).

From Eqs. (29) and (30), an ideal mounting system can be conceptualized in which the isolator impedance is infinite at low frequencies, and zero above the vehicle maneuvering bandwidth.[8] An isolator of this form would hold the engine and structure together at a zero relative displacement at low frequencies and provide perfect vibration isolation at high frequencies. Unfortunately, this ideal mounting system cannot be realized with passive isolators, and can only be approached by active isolation systems.[8]

Multidimensional Isolation Systems

In a single isolator mounting system, the magnitude of the effectiveness was used to quantify performance. In this section, the matrix equivalent of the scalar magnitude, the spectral norm, will be used to extend the basic concepts developed in the previous section for single isolator systems to multidimensional isolation systems. We will also see the role that directionality plays in the performance of multidimensional mounting systems.

The spectral, or Hilbert, norm of a complex matrix E is defined as the maximum singular value

$$\|E\|_s = \sigma_{\max}(E) = \sqrt{\lambda_{\max}(E^H E)} \tag{32}$$

where $\|E\|_s$ is the spectral norm of the matrix E, $\sigma_{\max}(E)$ is the largest singular value of E, E^H is the complex conjugate transpose of E, and $\lambda_{\max}$ is the largest eigenvalue of $E^H E$. Because the matrix E is a function of frequency, so is $\sigma_{\max}(E)$. The maximum singular value is a convenient measure of the maximum size of a matrix. A measure of the minimum size of a matrix, on the other hand, is the minimum singular value, defined as

$$\sigma_{\min}(E) = \sqrt{\lambda_{\min}(E^H E)} \tag{33}$$

where $\sigma_{\min}(E)$ is the minimum singular value of E, and $\lambda_{\min}$ is the minimum eigenvalue of $E^H E$. Because of numerical issues, the singular values of a matrix are usually obtained through a singular value decomposition, which is described in more detail in standard linear algebra textbooks such as Golub and Van-Loan.[9] The singular value decomposition is defined through the expression $E = \boldsymbol{U}\Sigma \boldsymbol{V}^H$, where $\boldsymbol{U}$ is defined as the left singular vector, $\boldsymbol{V}$ is defined as the right singular vector, and Σ is a matrix which contains the singular values on its diagonal elements. The singular values and singular vectors can easily be computed using standard matrix manipulation programs such as Ctrl-C.[10]

Taken together, the minimum and maximum singular values bound the magnitude of a multidimensional system, providing a frequency-dependent measure of performance. These bounds are necessary because, unlike the single isolator case, the exact performance of a multidimensional system cannot be predicted without explicit knowledge of the "directionality" aspects of the disturbance. For example, a precise prediction of performance in the isolated system would require measurements of the magnitude and phase of the unisolated velocities, or forces, at each isolator location when the engine was generating a vibratory disturbance. Unfortunately, these measurements are rarely available, especially in the design stage, when only the engine and structure impedances may be known.

Even though the directionality of the disturbance is generally unknown, the directions corresponding to the best and worst mount performances can be explored for MIMO systems using the minimum and maximum singular vectors. For example, the largest output will be generated along the direction of the maximum left singular vector when the input vector coincides with the direction of the maximum right singular vector. In this analysis, the input vector could represent the magnitude and phase of the isolated velocities, or forces, at each mounting location for a particular frequency of interest, and the output vector could represent the magnitude and phase of the unisolated velocities, or forces, at each mounting location for the same frequency of interest. If the right and left singular vectors are normalized to unity, then the magnitude of the output is equal to the maximum singular value. Also, the smallest output will be generated along the direction of the minimum left singular vector when the input vector coincides with the direction of the minimum right singular vector. If the right and left singular vectors are normalized to unity, then the magnitude of the output is equal to the minimum singular value. The minimum and maximum singular vectors therefore provide insight into the direction in which the best and worst performances will occur.

The vector norm of the ratio of the hard-mounted velocities to the isolated velocities can be bounded by the minimum and maximum singular values of the velocity effectiveness matrix

$$\sigma_{\min}(E_v) \leq \sqrt{\frac{(\boldsymbol{v}_s^h)^H \boldsymbol{v}_s^h}{(\boldsymbol{v}_s^i)^H \boldsymbol{v}_s^i}} \leq \sigma_{\max}(E_v) \tag{34}$$

where $\sigma_{min}(E_v)$ and $\sigma_{max}(E_v)$ are the respective minimum and maximum singular values of the velocity effectiveness matrix E_v. Equivalently, the maximum and minimum singular values of the force effectiveness matrix can be used to bound the norm of the transmitted forces in the hard-mounted and isolated mounting systems. One interesting observation is that although the eigenvalues of E_f and E_v are identical [because E_f and E_v are related through a similarity transformation defined in Eq. (24)], their singular values are not necessarily identical. This implies that, in general, the force and velocity effectiveness of a multidimensional mounting system will not be identical. The approach of using singular values to quantify the performance of multivariable systems has recently gained popularity in the multivariable control community.[11,12]

The tradeoffs examined in the previous section between minimal relative displacement at low frequency, and isolation at high frequency, can now be examined in terms of the singular values of the effectiveness matrix. The low-frequency minimal relative displacement requirement is satisfied when the maximum singular value of the effectiveness matrix is approximately equal to one. A high degree of isolation is achieved when the minimum singular value of the effectiveness matrix is much greater than one.

Now that we have seen how to use the singular values of the effectiveness matrix to characterize the multidimensional isolator performance, let's consider the isolator design problem. This will be done first in terms of force effectiveness using Eq. (27), and then in terms of velocity effectiveness using the similarity transformation defined in Eq. (24). At low frequencies, the force effectiveness should equal

$$E_f \approx I \tag{35}$$

or equivalently

$$Z_{eq}Z_i^{-1} \approx 0 \tag{36}$$

In terms of singular values, Eq. (36) is satisfied if

$$\sigma_{max}(Z_{eq}Z_i^{-1}) << 1 \tag{37}$$

The maximum singular value of the isolator impedance matrix can be related to the maximum singular value of the equivalent engine-structure impedance matrix using the following matrix norm inequality[12]:

$$\sigma_{max}(Z_{eq}Z_i^{-1}) \le \sigma_{max}(Z_{eq})\sigma_{max}(Z_i^{-1}) \tag{38}$$

Hence, Eq. (37) is automatically satisfied if

$$\sigma_{max}(Z_{eq})\sigma_{max}(Z_i^{-1}) << 1 \tag{39}$$

If the expression $\{\sigma_{max}(Z_i^{-1})\} = 1/[\sigma_{min}(Z_i)]$[12] is inserted into Eq. (39), we obtain

$$\sigma_{min}(Z_i) >> \sigma_{max}(Z_{eq}) \tag{40}$$

At high frequencies, the task of the isolator is to isolate the structure from engine vibration. In terms of the singular values of the force effectiveness matrix defined in Eq. (27), this simplifies to

$$\sigma_{min}(E_f) = \sigma_{min}(I + Z_{eq}Z_i^{-1}) >> 1 \tag{41}$$

Next, we will relate the minimum singular value of E_f to the minimum singular value of the matrix $Z_{eq}Z_i^{-1}$ using the matrix triangle inequality[12]

$$\sigma_{min}(Z) - 1 \le \sigma_{min}(I + Z) \le \sigma_{min}(Z) + 1 \tag{42}$$

When $\sigma_{min}(Z) >> 1$, the matrix triangle inequality in Eq. (42) can be approximated as

$$\sigma_{min}(Z) \approx \sigma_{min}(I + Z) \tag{43}$$

If we substitute the isolator effectiveness quantities shown in Eq. (41) into Eq. (43) we obtain

$$\sigma_{min}(Z_{eq}Z_i^{-1}) \approx \sigma_{min}(I + Z_{eq}Z_i^{-1}) >> 1 \tag{44}$$

Next, using the matrix norm inequality,[12] Eq. (44) becomes

$$\sigma_{min}(Z_{eq}Z_i^{-1}) \ge \sigma_{min}(Z_{eq})\sigma_{min}(Z_i^{-1}) \tag{45}$$

By combining Eq. (45) with the relation $\sigma_{max}(Z_i^{-1}) = 1/[\sigma_{min}(Z_i)]$, we see that the inequality given by Eq. (44) is automatically satisfied if

$$\sigma_{max}(Z_i) << \sigma_{min}(Z_{eq}) \tag{46}$$

For force isolation, Eqs. (40) and (46) are the multidimensional equivalents of the single isolator design relations shown earlier in Eqs. (29) and (30).

The similarity transformation defined in Eq. (24) relating force and velocity effectiveness can be used to extend the isolator design relations for force effectiveness to velocity effectiveness. From Eqs. (24) and (27), the velocity effectiveness matrix is equal to

$$E_v = Z_s^{-1}E_fZ_s = I + Z_s^{-1}Z_{eq}Z_i^{-1}Z_s \tag{47}$$

At low frequency, the velocity effectiveness matrix should approximately equal the identity matrix. From Eq. (47), we see that this occurs when

$$\sigma_{max}(Z_s^{-1}Z_{eq}Z_i^{-1}Z_s) << 1 \tag{48}$$

Using the matrix norm inequality defined earlier in Eqs. (38) and (45), we can show that Eq. (48) is automatically satisfied if

$$\sigma_{max}(Z_s^{-1})\sigma_{max}(Z_{eq})\sigma_{max}(Z_i^{-1})\sigma_{max}(Z_s) << 1 \tag{49}$$

Equation (49) can be simplified if we define a matrix condition number as

$$k(Z_s) = \sigma_{max}(Z_s^{-1})\sigma_{max}(Z_s) \tag{50}$$

Equation (49) then becomes

$$\sigma_{min}(Z_i) >> \sigma_{max}(Z_{eq})k(Z_s) \tag{51}$$

At high frequencies, the minimum singular value of the effectiveness matrix should be very large. In terms of the singular values of the velocity effectiveness matrix defined in Eq. (47), this requirement becomes

$$\sigma_{min}(E_v) = \sigma_{min}(I + Z_s^{-1}Z_{eq}Z_i^{-1}Z_s) >> 1 \tag{52}$$

By following the procedure outlined in Eqs. (41–46), we see that Eq. (52) is satisfied when

$$\sigma_{max}(Z_i) << \sigma_{min}(Z_{eq})k(Z_s) \tag{53}$$

Since the condition number $k(Z_s)$ is always greater than or equal to unity, a conservative estimate for isolator design that results in a high level of both force and velocity effectiveness at high frequencies is

$$\sigma_{max}(Z_i) << \sigma_{min}(Z_{eq}) \tag{54}$$

Equation (54) is the multidimensional equivalent to the single isolator design relation shown earlier in Eq. (30). However, one significant difference between the single and multiple isolator mounting systems is the importance of directionality in multiple isolator mounting systems. For example, because the maximum and minimum singular values represent a worst-case scenario, the relation given in Eq. (54) will never overestimate the performance of the mounting system. Equation (54) can, however, severely underestimate the performance of the mounting system if the difference between the minimum and maximum singular values of the equivalent engine and structure matrix is large, and the directionality of the disturbance coincides with the maximum right singular vector of the equivalent engine and structure matrix.

Analytical Example

This example demonstrates how to determine the velocity effectiveness matrix E_v. Also, the example shows that significant error is introduced when each isolator is treated as the only vibration transmission path by applying the scalar version of Eq. (23). The system for this example is shown in Fig. 3.

The engine and structure are represented by rigid masses that can move with velocities v_s^{i1}, v_e^{i1}, v_s^{i2}, and v_e^{i2}. Assuming appropriate units, the masses, lengths, stiffnesses, and damping coefficients are all equal to one, and the moments of inertia are equal to two. For this example, the (2 × 2) isolator impedance matrix is given by

$$Z_i = \begin{bmatrix} 1 + \dfrac{1}{s} & 0 \\ 0 & \dfrac{1}{s} \end{bmatrix} \tag{55}$$

where s is the Laplace variable. The frequency response function can easily be obtained from the Laplace transfer function by setting $s = j\omega$, where j is the complex number $\sqrt{-1}$, and ω is the radian frequency. The frequency response function and Laplace transfer function can conveniently be interchanged in the effectiveness expressions. In this example, the Laplace form is used because of its simplicity.

The impedance matrix of the uncoupled structure and engine are identical and can be determined from the equations of motion as

$$Z_s = Z_e = \frac{s}{4}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \tag{56}$$

Substituting Eqs. (55) and (56) into Eq. (23) yields the velocity effectiveness matrix E_v given by

$$E_v = \begin{bmatrix} \dfrac{3s^2 + 8s + 8}{8s + 8} & -\dfrac{s^2}{8s + 8} \\ -\dfrac{s^3 + s^2}{8s + 8} & \dfrac{3s^3 + 3s^2 + 8s + 8}{8s + 8} \end{bmatrix} \tag{57}$$

In practice, the actual disturbances are rarely known. However, in this example, a disturbance force $F_D(s)$ and disturbance torque $\tau_D(s)$ are shown in Fig. 3. When the engine is hard-mounted to the structure (the mounts possess an infinite impedance), the input disturbances cause the following hard-mounted velocities:

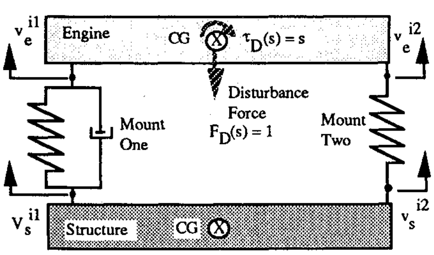

Fig. 3 Analytical example mounting system.

$$v_s^h = \begin{bmatrix} \dfrac{s + 2}{4s} \\ \dfrac{-s + 2}{4s} \end{bmatrix} \tag{58}$$

By inverting E_v in Eq. (57) and solving Eq. (22) for v_s^i, we obtain

$$v_s^i = \begin{bmatrix} \dfrac{3s^4 + 8s^3 + 16s^2 + 24s + 16}{4s(s^4 + 3s^3 + 6s^2 + 8s + 8)} \\ \dfrac{s^4 + 8s + 16}{4s(s^4 + 3s^3 + 6s^2 + 8s + 8)} \end{bmatrix} \tag{59}$$

Now, consider treating each mount as the only vibration transmission path. To do so, the scalar version of Eq. (23) is applied to each mount. This approach incorrectly neglects the coupling effects. For mount one, we have the following impedances:

$$Z_i = 1 + (1/s) \tag{60}$$

$$Z_s = Z_e = (2s/3) \tag{61}$$

Equation (61) assumes that the attachment point for mount two is free for both the engine and the structure. The scalar velocity effectiveness of mount one (E_{v1}) is given by

$$E_{v1} = \frac{s^3 + 3s^2 + 3s}{3(s + 1)} \tag{62}$$

For mount two, the following impedances apply:

$$Z_i = (1/s) \tag{63}$$

$$Z_e = Z_s = (2s/3) \tag{64}$$

Equation (64) assumes that the attachment for point one is free for both engine and structure. The scalar velocity effectiveness of mount two (E_{v2}) is given by

$$E_{v2} = [(s^2 + 2)/3] \tag{65}$$

Using element (1, 1) in Eq. (58) as the hard-mounted velocity v_s^{h1} and the scalar effectiveness of Eq. (60), the scalar version of Eq. (22) yields the isolated velocity:

$$v_s^{i1} = \frac{3}{4s^2}\frac{s^2 + 3s + 2}{s^2 + 3s + 3} \tag{66}$$

Similarly, using element (2, 1) in Eq. (58) as the hard-mounted velocity v_s^{h2} and the scalar effectiveness of Eq. (65), the scalar version of Eq. (22) yields the isolated velocity:

$$v_s^{i2} = \frac{3}{4s}\frac{-s + 2}{s^2 + 2} \tag{67}$$

The inaccuracy of the scalar calculations can be seen by comparing element (1, 1) of Eqs. (59–66), and by comparing element (2, 1) of Eqs. (59–67). Figure 4 graphically shows element (1, 1) of Eqs. (59) and (66).

Figure 5 shows element (2, 1) of Eqs. (59) and (67). Note that Eqs. (66) and (67) inaccurately predict the actual isolated velocities of Eq. (59). In particular, the high-frequency isolation performance is overestimated by the scalar method.

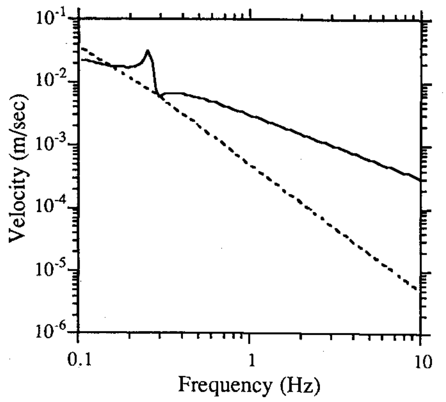

Fig. 4 Mount one velocity comparison. Multidimensional (solid line), single isolator (dashed line).

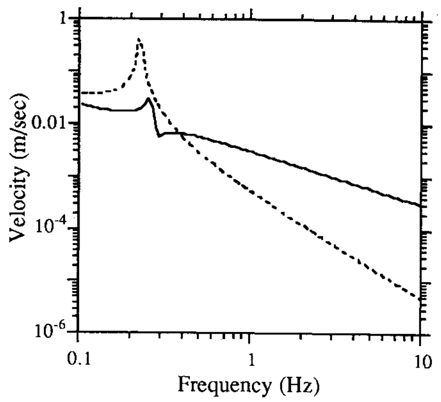

Fig. 5 Mount one velocity comparison. Multidimensional (solid line), single isolator (dashed line).

Experiment

An experiment, illustrated in Fig. 6, was also conducted to demonstrate the multidimensional mount effectiveness. An electromagnetic shaker applied disturbance forces to the engine-isolator-structure system. Two aluminum T-shaped beams, representing the engine and structure, were connected by isolators at three points. The engine, structure, and isolators used in this experiment are not necessarily representative of an actual aircraft engine mounting system. However, because of the presence of many lightly damped modes, the experiment was sufficiently complicated to verify the mount effectiveness method. Accelerance FRF measurements of the engine and structure were obtained at each isolator location by applying a band-limited (0–200 Hz) white noise force to the unconnected engine and structure, and measuring the resulting accelerations at each of the three mounting locations (for a more in-depth examination of the FRF measurement, see Bendat and Piersol,[13] or Ewins[14]). Accelerance relates an output acceleration to an input force and can be converted to impedance through the following formula:

$$Z = j\omega A^{-1} \tag{68}$$

where Z is the impedance matrix, j is the complex variable $\sqrt{-1}$, ω is the radian frequency, and A is the measured accelerance matrix. The engine and structure drive point impedances at mount location one are shown in Figs. 7 and 8, respectively. For systems with a large number of isolators, or in the region of the resonances of the structure, inconsistencies in measured data can cause the matrix inversion defined in Eq. (68) to become ill-conditioned.[7] The engine and structure used in this experiment contain lightly damped resonances that are typical of those found in lightly damped, distributed structures. However, in this experiment, no difficulties with singularities were experienced due to the spectral resolution (0.5 Hz) and the low number of measurement locations (three). Furthermore, Leuridan and Aquilina[7] outlined an approach where the singular value decomposition can be used to generate the impedance matrix for ill-conditioned systems.

Three identical sandwich mounts, consisting of natural rubber bonded between two metal plates, were used as the isolators. The impedance of each isolator was measured by blocking one end of the isolator, applying a band-limited (0–200 Hz) white noise velocity to the other end, and measuring the resulting force across the mount. The magnitude of a measured isolator impedance is shown in Fig. 9. For the hard-mounted (unisolated) system, a thin steel rod connected the engine and structure. The rod was very stiff axially, while soft in bending and torsion, and emulated a "stinger" that is commonly used

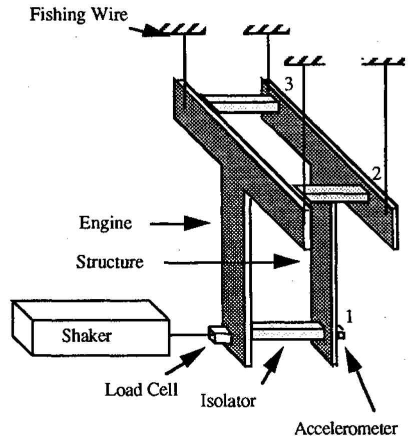

Fig. 6 Multidimensional effectiveness experiment.

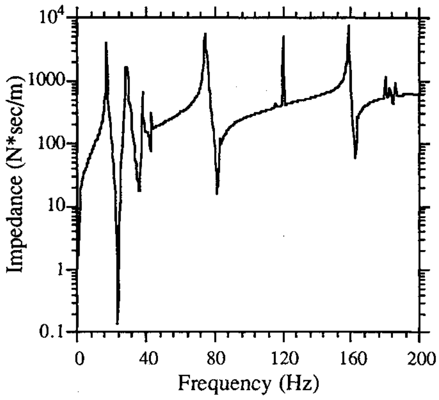

Fig. 7 Engine impedance magnitude.

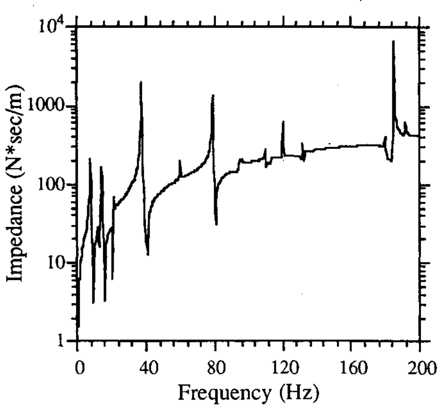

Fig. 8 Structure impedance magnitude.

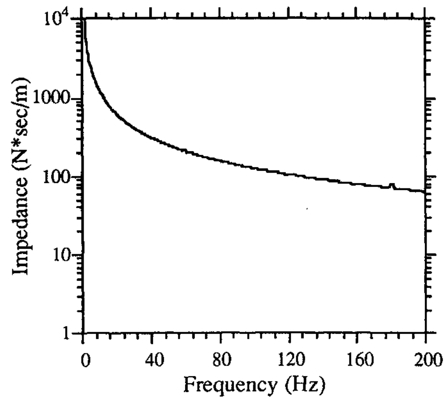

Fig. 9 Isolator impedance magnitude.

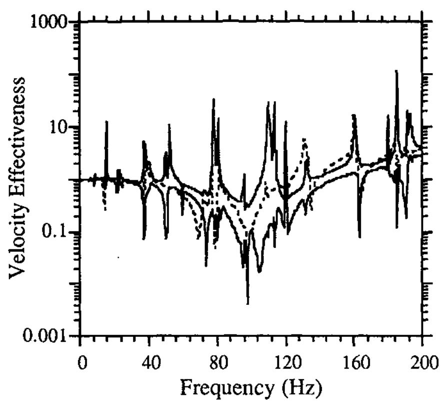

Fig. 10 Effectiveness bounds. Singular values (solid line), measured norm (dashed line).

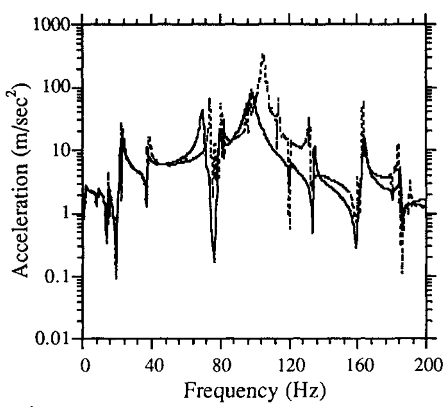

Fig. 11 Acceleration comparison. Measured acceleration (solid line), predicted acceleration (dashed line).

for extracting FRF measurements. The rod minimized the torque at the engine-structure interface.

The measured engine and structure accelerances, as well as the measured isolator impedance, were then transferred to a mainframe computer. With the help of the matrix manipulation program Ctrl-C,[10] the engine and structure accelerances were converted to impedance matrices using Eq. (68). The velocity effectiveness matrix defined in Eq. (23) was also evaluated using Ctrl-C.

Figure 10 shows a comparison between the upper and lower singular values of the velocity effectiveness matrix, and the measured norm of the velocity response ratio. The maximum and minimum singular values of the effectiveness matrix provide an estimate of the degree of velocity isolation for the entire mounting system. When the minimum singular value of the effectiveness matrix is greater than unity, the mounting system provides isolation by decreasing the structural vibration. When the maximum singular value is less than unity, the mounting system amplifies the structural vibration. The measured norm ratio of the isolated to hard-mounted accelerations fell acceptably within the upper and lower bounds for nearly all frequencies. For this system, isolation is provided by the mounting system above approximately 140 Hz. Below 140 Hz, the response of the system is worse than the equivalent hard-mounted system.

Figure 11 compares the predicted acceleration with the measured acceleration for the isolated mounting system at mounting location one. The predicted and measured results compare closely. Any differences can probably be attributed to either measurement error or isolator nonlinearity.

Conclusions

A multidimensional isolator effectiveness method was derived and presented. The effectiveness matrix was constructed using frequency response functions of the engine, isolator, and structure. Because measurements of the engine, isolator, and structure frequency response functions can be used to estimate the performance of the mounting system, system identification techniques such as modal analysis are avoided. This is a tremendous advantage for both modally dense and heavily damped systems.

The maximum and minimum singular values of the effectiveness matrix were used to show direct analogies between the single isolator effectiveness and the multidimensional effectiveness. It was shown that the isolation system was effective when the maximum singular value of the isolator impedance matrix was much less than the minimum singular value of the equivalent engine and structure impedance matrix. The directionality of the disturbance was also shown to be important in predicting performance for multiple mount isolation systems.

An analytical example and experiment were used to demonstrate the method. The analytical example showed that a multidimensional effectiveness must be used for systems with a high degree of coupling. The experimental results showed good agreement between the predicted and measured structural motions.

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